ARTICLES

Critical hysteresis for *n*-component magnets

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Earlier work on dynamical critical phenomena in the context of magnetic hysteresis for uniaxial (scalar) spins is extended to the case of a multicomponent (vector) field. From symmetry arguments and a perturbative renormalization-group approach (in the path-integral formalism), it is found that the generic behavior at long time and length scales is described by the scalar fixed point (reached for a given value of the magnetic field and of the quenched disorder), with the corresponding Ising-like exponents. By tuning an additional parameter, however, a fully rotationally invariant fixed point can be reached, at which all components become critical simultaneously, with $O(n)$-like exponents. Furthermore, the possibility of a spontaneous nonequilibrium transverse ordering, controlled by a distinct fixed point, is unveiled and the associated exponents calculated. In addition to these central results, a didactic ''derivation'' of the equations of motion for the spin field are given, the scalar model is revisited and treated in a more direct fashion, and some issues pertaining to time dependences and the problem of multiple solutions within the path-integral formalism are clarified.
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I. INTRODUCTION

In a great variety of nonequilibrium situations, critical behavior is observed as a system evolves from one of its possible states to another. Some examples are charge-density waves, fluctuating interfaces and lines, cracks and fractures, and the Barkhausen effect in magnets. These systems evolve, respectively, from a state without current to a state with current, from a stationary to a moving state, and from a connected to a ruptured state, from a downward to an upward magnetization. Under specific conditions (i.e., preparation of the system), the transition between the two states is critical (or continuous), exhibiting diverging correlation lengths and scaling laws. The qualitative descriptions of the dynamics of the different physical situations mentioned are very similar in the key parameters and mechanisms which govern criticality. The quantitative descriptions are also close, in that the dynamics of motion can be described by continuum field equations, and share many common features. In this paper we focus on magnetic systems, more specifically on a lattice of spins with ferromagnetic exchange (coupling). While our qualitative and quantitative analyses will be in this framework, some aspects of the discussion may apply to other systems, such as depinning transition of flux lines or fractures in disordered media.

What do we mean by ''the key parameters and mechanisms which govern criticality''? Consider a ferromagnet in an external magnetic field which increases slowly from $-\infty$ to $+\infty$. Each spin feels a local field equal to the average of the surrounding spins multiplied by the coupling constant (JM_i in the case of the ith spin), plus the external magnetic field H. At $H=-\infty$, all the spins point ''downward'' ($M_i=-1$ for unit spins), thus $JM_i=-J$ initially. At zero temperature, each spin simply points in the direction of the local field JM_i+H, and so none of the spins change before H reaches J, at which point they all flip upwards. The magnetization M thus jumps from -1 to $+1$ at $H=J$. This scenario for a perfectly clean system is modified by introducing some disorder. At each lattice point occupied by a spin, add a random field, h_i, to the local field, JM_i+H. (Specifically, let h_i be an uncorrelated random variable, chosen from a Gaussian distribution centered at zero.) Then, the spins flip in a much less coherent way: as soon as JM_i+H+h_i becomes positive, the ith spin flips. The upward h_i's enhance the increase in magnetization for low H, whereas the downward h_i's suppress it for high H. This results in the reduction of the magnitude of the jump in magnetization. Clearly, if we broaden the random field distribution, i.e., increase the amount of disorder, the discontinuity in M is further suppressed, until the curve $M(H)$ eventually becomes smooth for a high enough disorder (Fig. 1). We can imagine a sequence of hysteresis curves, corresponding to a succession of increasing amounts of disorder, say for the variance of the random field, $\overline{h^2}$, going from zero to infinity. The curve displays a discontinuity for small $\overline{h^2}$, and is smooth for large $\overline{h^2}$. The transition between the two regimes occurs at a critical point reminiscent of continuous or second-order phase transitions [1–4]: at the critical amount of disorder the discontinuity collapses to a point at which the slope is infinite. This is referred to as a *critical hysteresis*, for which, at a given magnetic field, the susceptibility diverges. The amount of disorder and the magnetic field are the two parameters we have to tune to observe criticality.

In the past few years, disorder-induced critical hysteresis in magnets has been the subject of much interest [1–6]. Dahmen, Sethna, and others studied this problem via a mean-field approximation, one-loop momentum-space renormal-

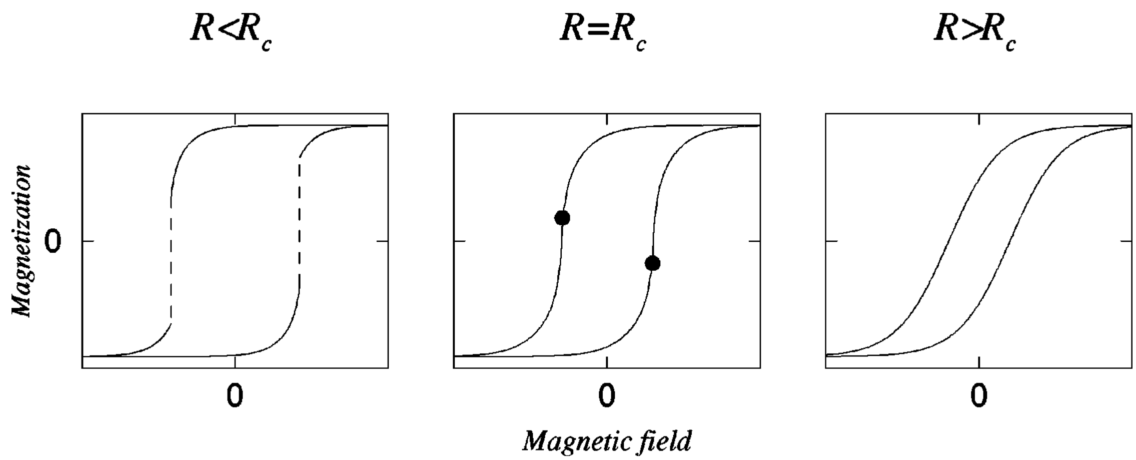

FIG. 1. Schematic hysteresis curves for different values of the disorder R. Left: $R<R_c$ (discontinuous hysteresis); center: $R=R_c$ (critical hysteresis); right: $R>R_c$ (smooth hysteresis). In each case, the lower (upper) curve corresponds to an increasing (decreasing) magnetic field.

ization, and numerical simulations. Also, they describe a mapping of this nonequilibrium problem onto the equilibrium random field Ising model, which can in turn be mapped (close to the upper critical dimension) onto the pure Ising model in two lower dimensions [7]. Throughout their work, they consider a scalar order parameter. They study the dynamics of an Ising (or discrete) spin field driven by an increasing magnetic field and in the presence of a random field, at zero temperature.

The question we ask here is the following: How are the phase diagram and critical behavior modified if the order parameter is vectorial instead of scalar? More precisely stated: in the renormalization-group (RG) framework, is the fixed point which controls the above-mentioned hysteretic criticality one and the same for both the Ising and the vectorial cases? And if not, how do the exponents differ? In equilibrium, the disordering of scalar and vector systems is described by distinct universality classes [7]. Also, in the closely related context of depinning transitions, the distinction between interfaces (scalar) and flux lines (vector) was noted in Ref. [8].

The answer can readily be guessed on symmetry grounds. Indeed, symmetry considerations lead to two distinct cases. In the first, *and generic one*, the critical hysteresis curve is such that the susceptibility diverges at a nonvanishing value of either the magnetic field or the magnetization. Then, although we consider continuous spins, the full rotational symmetry of the problem is broken at the critical point, a unique preferred direction is picked, and Ising-like critical behavior results. It is similarly argued, in Ref. [2], Appendix E, and Ref. [3], Appendix L, that the universality class of the random field scalar model extends also to random bond scalar models (with a positive nonzero mean of the bonds' values) and to random anisotropy $O(n)$ models. On the other hand, if both the magnetic field and the magnetization vanish when the susceptibility diverges, we may have a fully rotation invariant system at the critical point. In that case, we expect $O(n)$-like criticality, with exponents that differ from those of the Ising model. Evidently, a vanishing magnetization at $H=0$ is not a sufficient condition for a full rotational invariance. In particular, due to its history, the system might well display higher-order anisotropies, such as, e.g., $\overline{s_\parallel^2}\neq\overline{s_{\perp\alpha}^2}$, where $s_\parallel$ is the component of the spin field parallel to $\mathbf{H}$ and $s_{\perp\alpha}$ is any parpendicular component. This issue is resolved by a renormalization-group analysis, which confirms our various guesses. Furthermore, it discloses the possibility of a ''transverse critical point,'' corresponding to an instability of the magnetization component *perpendicular* to the external magnetic field.

The present paper is organized as follows. In Sec. II, we construct the equation of motion of a vector spin field. A path-integral formalism is described in Sec. III, with which the problem of renormalizing the equation of motion is recast into that of renormalizing a partition function. The time dependences, as well as the subtleties associated with ''many energy minima,'' are also examined. In Sec. IV, the renormalization-group treatment of the problem is presented. First, we define the coordinates' and fields' rescalings, and calculate the free propagator. Then, we discuss successively the scalar and vector models. For the latter, the different cases (hysteretic or nonhysteretic, longitudinal or transverse criticality) are analyzed, and the corresponding recursion relations and exponents are obtained.

II. EQUATIONS OF MOTION

The equations of motion for a scalar field are discussed in Refs. [1–3], and their generalization to a multicomponent field is straightforward. Nonetheless, for completeness and to emphasize our perspective, we present here a didactic introduction to the equations of motion for vectorial spins, at zero temperature [9]. Consider a d-dimensional lattice, with a spin $\mathbf{s}_i\in\Re^n$ at each site i, subject to a magnetic field which changes slowly from $-\infty$ to $+\infty$, say linearly in time, $\mathbf{H}=\mathbf{\Omega}t$. The rate $\mathbf{\Omega}$ can be made arbitrarily small in magnitude and points along the first axis of our coordinates, i.e., $H_1=\Omega t\equiv H$, $H_2=\cdots=H_n=0$. The time-dependent magnetic field implies a time-dependent energy function $\mathcal{H}(t)$. At zero temperature $(T=0)$, the spins simply follow the local minimum of this energy function according to

$$\eta\partial_t\mathbf{s}_i=-\frac{\delta\mathcal{H}}{\delta\mathbf{s}_i},\qquad(1)$$

starting from a uniform downward pointing configuration at $t=-\infty$. The parameter η controls the relaxation rate of spins towards the time-dependent local energy minimum of $\mathcal{H}$. The smaller η is, the faster spins relax, and the less they lag behind the energy minimum [10].

The following ''key features'' guide us in constructing the Hamiltonian $\mathcal{H}$. First, to describe a ferromagnet, we include in $\mathcal{H}$ couplings J_{ij} which tend to align the spins. In addition to the external uniform field $\mathbf{H}$ which drives the system, we include quenched random fields $\mathbf{h}_i$. The $\mathbf{h}_i$'s are uncorrelated Gaussian random variables, chosen from the distribution

$$\rho[\mathbf{h}]=N\exp\left(-\sum_i \frac{\mathbf{h}_i^2}{2R}\right), \tag{2}$$

where N is a normalization factor. For calculational convenience we shall work with soft spins (whose magnitude can take any real value), which can be thought of as a coarse-grained picture of a hard spin field. To avoid the unphysical instability of spins diverging in magnitude, we introduce an on-site potential $V(\mathbf{s}_i)$, which constrains the magnitude to remain close to 1 (or some finite number). The potential V is spherically symmetric (a ''double well'' in the scalar model and a ''Mexican hat'' in the vector model) and is expressed through its Taylor expansion about the origin, as

$$V(\mathbf{s}_i)=-\frac{c_1}{2}\mathbf{s}_i^2-\frac{c_2}{4}(\mathbf{s}_i^2)^2+\cdots. \tag{3}$$

Whether or not V is analytic at the origin is unimportant, since $|\mathbf{s}_i|$ is constrained to be close to 1 (and not to 0). The full Hamiltonian is now given by

$$\mathcal{H}=-\frac{1}{2}\sum_{i,j} J_{ij}\mathbf{s}_i\cdot\mathbf{s}_j+\sum_i[-\mathbf{H}\cdot\mathbf{s}_i-\mathbf{h}_i\cdot\mathbf{s}_i+V(\mathbf{s}_i)]. \tag{4}$$

The gradient descent with this Hamiltonian leads to the equation of motion

$$\eta\partial_t\mathbf{s}_i=\sum_j J_{ij}\mathbf{s}_j+\mathbf{H}+\mathbf{h}_i-\frac{\partial V}{\partial \mathbf{s}_i}. \tag{5}$$

Assuming that J_{ij} is a function of the separation between spins results, in the continuum limit, in

$$\eta\partial_t\mathbf{s}(\mathbf{x})=\int d^dx'\,J(\mathbf{x}-\mathbf{x}')\mathbf{s}(\mathbf{x}')+\mathbf{H}(t)+\mathbf{h}(\mathbf{x})-\frac{\partial V}{\partial \mathbf{s}}. \tag{6}$$

While rewriting the problem in the continuum limit, we must impose some limit on how fine-grained the spin field $\mathbf{s(x)}$ may be because of its lattice origin. In other words, $\mathbf{s(x)}$ is a superposition of Fourier components whose wave numbers are restricted from zero to some cutoff Λ.

Finally, if the exchange function decays fast enough (as its argument increases) for its Fourier component to be nonsingular at the origin of momentum space, i.e., if $\tilde{J}(\mathbf{q})=1-Kq^2+\cdots$, then

$$\int d^dx'\,J(\mathbf{x}-\mathbf{x}')\mathbf{s}(\mathbf{x}')\approx\mathbf{s}(\mathbf{x})+K\nabla^2\mathbf{s}(\mathbf{x}), \tag{7}$$

and the equation of motion can be written as

$$\eta\partial_t\mathbf{s}=K\nabla^2\mathbf{s}+\mathbf{H}+\mathbf{h}-\frac{\partial\tilde{V}}{\partial\mathbf{s}}, \tag{8}$$

where the coefficient c_1 in the the expansion of the potential has been modified in order to take the $\mathbf{s(x)}$ term of Eq. (7) into account.

III. PATH-INTEGRAL FORMALISM

In order to identify the critical properties of our model, we should ideally solve its equation of motion. In practice, we study the behavior of Eq. (8) under a coarse-graining transformation. This allows us to locate and characterize a scale-invariant point. As a first step, we recast the equation of motion into a path integral (or generating functional), which incorporates the whole history of the system. The generating functional is written as the sum over all paths of the exponential of some action, which is then renormalized perturbatively. The advantage of reformulating the problem in this way is that we can express the perturbative treatment in a diagrammatic fashion similar to other field theories.

We define the generating functional [11] simply as the sum over all paths of a δ function, which makes each spin follow the time evolution given by Eq. (8), i.e.,

$$\begin{aligned}Z&=\int\mathcal{D}s\,\delta[\mathbf{s}-\text{solution of Eq. (8)}]\\&=\int\mathcal{D}s\,\delta\left(-\eta\partial_t\mathbf{s}+K\nabla^2\mathbf{s}+\mathbf{H}+\mathbf{h}-\frac{\partial\tilde{V}}{\partial\mathbf{s}}\right)\times(\text{Jacobian}),\end{aligned} \tag{9}$$

where $\mathcal{D}s$ stands for $\Pi\{$over ''all t,'' ''all $\mathbf{x}$,'' $\alpha=1,\ldots,n\}ds_\alpha(\mathbf{x},t)$. The Jacobian merely normalizes the value of Z to unity, and we shall henceforth ignore it [12]. Let us rewrite the δ function in its representation as the integral of an exponential $[2\pi\delta(f)=\int d\hat{s}\exp(i\hat{s}f)]$. Then, after absorbing a factor i in a redefinition of $\hat{\mathbf{s}}$, and dropping an infinite multiplicative constant (along with the Jacobian), we have

$$\begin{aligned}Z&=\int\mathcal{D}\hat{s}\mathcal{D}s\\&\quad\times\exp\left\{\int dt\,d^dx\,\hat{\mathbf{s}}\cdot\left(-\eta\partial_t\mathbf{s}+K\nabla^2\mathbf{s}+\mathbf{H}+\mathbf{h}-\frac{\partial\tilde{V}}{\partial\mathbf{s}}\right)\right\}\\&\equiv\int\mathcal{D}\hat{s}\mathcal{D}s\,\exp(\mathcal{S}).\end{aligned} \tag{10}$$

The *generating functional Z* enables us to evaluate all correlation and response functions. For example, the solution of Eq. (8) is

$$\mathbf{s}^{\rm sol}(\mathbf{x},t)=\int\mathcal{D}\hat{s}\mathcal{D}s\;\;\mathbf{s}(\mathbf{x},t)\exp(\mathcal{S}), \tag{11}$$

and its response to the magnetic field is given by

$$\frac{\delta s_1^{\rm sol}(\mathbf{x},t)}{\delta H(t')}=\int \mathcal{D}\hat{s}\mathcal{D}s\int d^d\mathbf{x}' \;\; \hat{s}_1(\mathbf{x}',t')s_1(\mathbf{x},t)\exp(\mathcal{S}). \tag{12}$$

Also, we can change the origin of time by a trivial reparametrization of the magnetic field, as, e.g., in

$$\mathbf{s}^{\rm sol}(\mathbf{x},t+\epsilon;\mathbf{H}(t))=\mathbf{s}^{\rm sol}(\mathbf{x},t;\mathbf{H}(t+\epsilon)). \tag{13}$$

The latter expression takes the form $\mathbf{s}^{\rm sol}(\mathbf{x},t;\mathbf{H}(t)+\mathbf{\Omega}\epsilon)$ if H is increased linearly in time at a rate Ω, whence

$$\mathbf{s}^{\rm sol}(\mathbf{x},t+\epsilon)-\mathbf{s}^{\rm sol}(\mathbf{x},t)=\int \mathcal{D}\hat{s}\mathcal{D}s\mathbf{s}(\mathbf{x},t)\exp(\mathcal{S})\times\left\{\exp\left(\int dt'd^dx'\hat{s}_1\Omega\epsilon\right)-1\right\}, \tag{14}$$

from which it follows that the dynamic susceptibility is calculated as

$$\frac{\partial \mathbf{s}^{\rm sol}(\mathbf{x},t)}{\partial t}=\Omega\int dt'd^dx'\int \mathcal{D}\hat{s}\mathcal{D}s\hat{s}_1(\mathbf{x}',t')\mathbf{s}(\mathbf{x},t)\exp(\mathcal{S}). \tag{15}$$

Since we are interested in the average of the correlations and responses over the random field, from now on we deal with the average of Z. This enables us to forget the stochastic variable h_α, trading it for a new term in the ''averaged action.'' Taking advantage of the Gaussian nature of h_α,

$$\overline{Z}=\int \mathcal{D}\hat{s}\mathcal{D}s \times\exp\left\{\int dtd^dx\hat{\mathbf{s}}\cdot\left(-\eta\partial_t\mathbf{s}+K\nabla^2\mathbf{s}+\mathbf{H}-\frac{\partial\tilde{V}}{\partial\mathbf{s}}\right)\right\} \times\overline{\exp\int dtd^dx\hat{\mathbf{s}}\cdot\mathbf{h}} \equiv\int \mathcal{D}\hat{s}\mathcal{D}s\exp(S), \tag{16}$$

with [using Eq. (2)]

$$S=\int dtd^dx\hat{\mathbf{s}}\cdot\left(-\eta\partial_t\mathbf{s}+K\nabla^2\mathbf{s}+\mathbf{H}-\frac{\partial\tilde{V}}{\partial\mathbf{s}}\right)+\int dtdt'd^dx\frac{R}{2}\hat{\mathbf{s}}(\mathbf{x},t)\cdot\hat{\mathbf{s}}(\mathbf{x},t'). \tag{17}$$

We have reformulated the theory, originally described by a dynamical differential equation, in terms of an action $S[\mathbf{s}(\mathbf{x},t)]$, which depends on the entire history of all spins. Thus S is a functional of the path which the system follows; the probability weight exp(S) picks the physical path and averages it over disorder. We can then study the symmetries and renormalization of the theory, as for equilibrium field theories.

The motion comprises two time scales: η and $(dH/dt)^{-1}$. The behavior of the system depends of course on the ratio of the two, and not on their respective values. For the calculation of static exponents, we let $\eta\to0$. Consider, for example, the exponent ν, with which the correlation length diverges. A diverging correlation length gives rise to an infinite susceptibility, detected by a nonvanishing response (of the magnetization) to an infinitesimal increase of the magnetic field. Clearly, such a behavior is obtained in our problem only if the magnetic field increases infinitely slowly [i.e., $(dH/dt)\to0$], or equivalently if $\eta\to0$. As η is a measure of how much the system lags behind its local energy minimum, for $\eta\to0$ the system always stays at the time-dependent minimum (as seen by setting $\eta=0$ in the equation of motion). In other words, the spin avalanches resulting from a small change in H spread instantaneously.

In the $\eta\to0$ limit, time evolution of the spin field is simply motion with the minimum in the energy landscape. There is nevertheless a subtlety involved as to the choice of the minimum. For example, because of the double-well shape of the potential, a scalar spin has the choice, during some time interval of its history, between two positions, both of which locally minimize the Hamiltonian. If we solve the equation of motion, there is no ambiguity, since we specify one of the energy minima as the initial condition, and then follow its time evolution. In particular, the initial condition corresponding to the case of a magnetic field which increases from $-\infty$ to $+\infty$ is one in which all the spins occupy the left minimum of their double wells [14]. In the path-integral formalism, however, no initial condition is specified. The weight e^S picks all possible solutions, corresponding to different initial conditions. Let us illustrate this point with, as in Ref. [15], the zero-dimensional nonrandom version of our model, defined by the equation of motion

$$\eta\partial_t s=-\frac{\partial}{\partial s}(-Hs+as^2+bs^4), \tag{18}$$

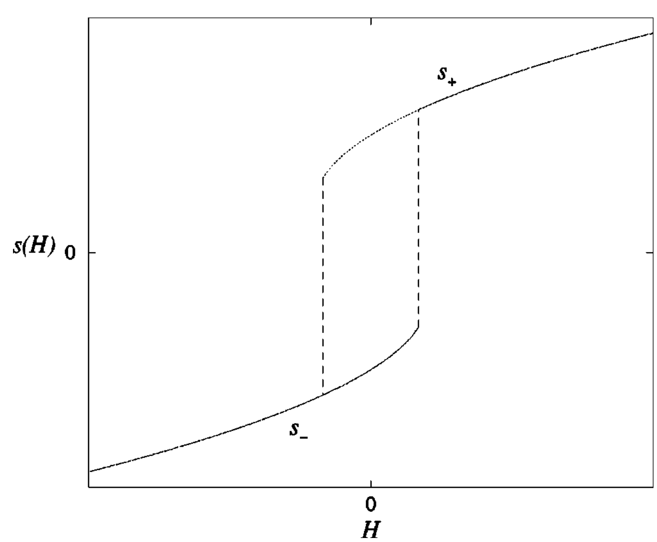

FIG. 2. Metastable solutions of the single-spin system. The physical solution for an increasing magnetic field (solid line) follows the lower curve (s_-) as long as it is present, then jumps along the dashed line to the upper curve (s_+).

with $\eta\to 0$. We can imagine s as a bead sitting at the minimum of the quartic potential $Q(s)=-Hs+as^2+bs^4$, and moving with it. With $a<0$ and $b>0$,$Q(s)$ has a single minimum if $|H|$ is larger than some value Δ, and two minima if $|H|$ is smaller than Δ. For H very negative, the bead sits at the single minimum of the curve, which becomes the left minimum when H is between $-\Delta$ and Δ. At $H=\Delta$, the left minimum disappears and the bead moves to the right minimum. Thus, in particular, $s(-H)\neq -s(H)$ and $s(H=0)\neq 0$: the motion of s *is hysteretic* (Fig. 2).

The action for this model is

$$S=\int dt\hat{s}(-\eta\partial_t s+H-as-bs^3). \tag{19}$$

Let $\langle s\rangle$ be the average of s with respect to the weight e^S. Calculating $\langle s\rangle$ perturbatively, it is easily seen (by counting the possible occurrences of s and $\hat{s}$ in a diagram) that each diagram comes with an odd number of H's. Therefore, $\langle s(-H)\rangle=-\langle s(H)\rangle$, and in particular, $\langle s(H=0)\rangle=0$, in apparent contradiction with the above solution. But as mentioned earlier, the partition function is merely the integral of a δ function which imposes the equation of motion. In the present case, it imposes

$$\frac{\partial}{\partial s}(-Hs+as^2+bs^4)=0. \tag{20}$$

Thus, in calculating $\langle s\rangle$, we pick all minima of the quartic form. In other words, $\langle s\rangle$ is the sum of two terms, $\langle s\rangle=\langle s_-\rangle+\langle s_+\rangle$, corresponding to the left and right minima. The physical ''bead'' solution is equal to $\langle s_-\rangle$ up to $H=\Delta$ and then $\langle s_+\rangle$ for H larger than Δ. Similarly, in the case of our original problem, the quantity $\mathbf{s}^{\rm sol}(\mathbf{x},t)$ of Eqs. (11) and (12) does not coincide with the physical solution we are looking for. In addition to the latter, $\mathbf{s}^{\rm sol}(\mathbf{x},t)$ contains other unwanted solutions corresponding to additional energy minima. We shall come back to this difficulty and circumvent it in the next section, both for the scalar and the vector models.

IV. PERTURBATIVE RENORMALIZATION

A. Coordinates and fields rescalings; the free propagator

Our renormalization-group transformation consists of the usual three steps. First, we coarse-grain the system, i.e., integrate out the modes with wave number between Λ/b and $\Lambda(b>1)$. Second, we rescale coordinates (i.e., change our length and time units), by setting

$$\mathbf{x}\to b\mathbf{x},\quad t\to b^z t, \tag{21}$$

or, equivalently,

$$\mathbf{q}\to b^{-1}\mathbf{q},\quad \omega\to b^{-z}\omega. \tag{22}$$

With this change of units, the coarse-grained fields vary on the same length scales as the original ones, and the lattice cutoff is preserved. Third, we rescale the fields according to

$$\mathbf{s}(b\mathbf{x},b^z t)\to\zeta\mathbf{s}(\mathbf{x},t),\quad \hat{\mathbf{s}}(b\mathbf{x},b^z t)\to\hat{\zeta}\hat{\mathbf{s}}(\mathbf{x},t), \tag{23}$$

or, equivalently,

$$\begin{aligned}\mathbf{s}(b^{-1}\mathbf{q},b^{-z}\omega)&\to b^{d+z}\zeta\mathbf{s}(\mathbf{q},\omega),\\ \hat{\mathbf{s}}(b^{-1}\mathbf{q},b^{-z}\omega)&\to b^{d+z}\hat{\zeta}\hat{\mathbf{s}}(\mathbf{q},\omega).\end{aligned} \tag{24}$$

With the choice

$$z=2, \tag{25}$$

$$\zeta=b^{2-d/2}, \tag{26}$$

$$\hat{\zeta}=b^{-2-d/2}, \tag{27}$$

the time derivative and Laplacian terms of the action, as well as the $\hat{s}\hat{s}$ terms, become scale invariant. The recursion relations will be calculated to the lowest nontrivial order in the interaction. To this order, no corrections of the parameters η,K, or R of the action in Eq. (17) occur in the coarse-graining transformation, i.e., η, K, and R are invariant under our perturbative renormalization.

As in the momentum-space RG treatment of the ϕ^4 theory, we consider the quadratic part of the action as a Gaussian (free) theory, and the rest as a perturbation (interaction). Following Refs. [1–3], for calculational convenience we treat the disorder-induced $\hat{s}\hat{s}$ term as an interaction, instead of including it in the Gaussian part. The free theory thus consists only of the $\hat{s}s$ part of the action, and the corresponding bare propagators are

$$\langle s_\alpha(\mathbf{q},\omega)s_\beta(\mathbf{q}',\omega')\rangle_0=0,$$

$$\langle\hat{s}_\alpha(\mathbf{q},\omega)\hat{s}_\beta(\mathbf{q}',\omega')\rangle_0=0,$$

$$\langle\hat{s}_\alpha(\mathbf{q},\omega)s_\beta(\mathbf{q}',\omega')\rangle_0=\delta_{\alpha\beta}2\pi\delta(\omega+\omega')(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\times\frac{1}{-\eta i\omega+Kq^2-r_\alpha}, \tag{28}$$

where $\langle\,\rangle_0$ denotes an average with respect to the Gaussian weight and the indices run from 1 to n. The parameter r_α is the q-independent coefficient of the quadratic $\hat{s}_\alpha s_\alpha$ term in the action. Fourier transforming back in time, we have

$$\langle\hat{s}_\alpha(\mathbf{q},t)s_\beta(\mathbf{q}',t')\rangle_0=\begin{cases}0 & \text{if}\quad t'\le t,\\ \delta_{\alpha\beta}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\exp\left[-\frac{Kq^2-r_\alpha}{\eta}(t'-t)\right]\Big/\eta & \text{if}\ t'>t.\end{cases} \tag{29}$$

(This is calculated for $r_\alpha<0$. As we shall see, r_α is indeed negative at criticality.) In the $\eta\to 0$ limit, the propagator becomes

$$\langle \hat{s}_\alpha(\mathbf{q},t) s_\beta(\mathbf{q}',t')\rangle_0 = \delta_{\alpha\beta}(2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}') \times \frac{1}{Kq^2-r_\alpha}\delta(t'-t^+). \tag{30}$$

That is, the contraction of $\hat{s}_\alpha(t)$ and $s_\alpha(t')$ is nonvanishing only if the two times are equal (actually, only if t' is infinitesimally higher than t). With this propagator, it is easily seen diagrammatically that, although the disorder-induced $\hat{s}\hat{s}$ terms couple different times, a renormalized vertex at time t is a function only of the other vertices *at the same time t.* Thus, slices of the action at different times renormalize independently from each other, and flow to their respective fixed points. This justifies the procedure of Refs. [1–3] of setting H constant for the calculation of static exponents. [In what follows, we shall need to correct the action of Eq. (17) to take care of the problem of multiple solutions. We shall, for example, expand S about a uniform but time-dependent value of the field, say some function $\sigma(t)$. Hence the vertices (coefficients in S), and in particular the ''masses'' r_α, become functions of the parameter $\sigma(t)$. Note that the above analysis of the $\eta\to 0$ limit is then legitimate only if $\sigma(t)$ is continuous, so that $\sigma(t^+)=\sigma(t)$. Below, we shall define our σ's in terms of the magnetization or analogous quantities. Thus, our analysis holds if we approach criticality from the high disorder side.]

B. The scalar model revisited

The zero-dimensional model discussed in Sec. III is nothing but the single-spin equivalent to the scalar field. The apparent contradiction mentioned therein is also present in the full model. Let us call $s_-(\mathbf{x},t)$ and $s_+(\mathbf{x},t)$ the solutions of Eq. (8), for a magnetic field H increasing from $-\infty$ to $+\infty$, or decreasing from $+\infty$ to $-\infty$, respectively. The magnetization measured experimentally is the average over space, or equivalently over the random field, of the above solutions,

$$m_\pm(H(t)) = \overline{s_\pm(\mathbf{x},t)}. \tag{31}$$

Generically, as is inferred from the single-spin case and observed experimentally, the magnetization displays hysteresis. In particular, $m_\pm(-H)\neq -m_\pm(H)$ and $m_\pm(H=0)\neq 0$. On the other hand, the action of Eq. (17) is invariant under the transformation $(\hat{s},s,H)\to(-\hat{s},-s,-H)$. This implies that the average of $s^{\rm sol}$ [Eq. (11)] satisfies $\overline{s^{\rm sol}(-H)} = -\overline{s^{\rm sol}(H)}$ and $\overline{s^{\rm sol}(H=0)}=0$. As explained above, $s^{\rm sol}$ contains unphysical contributions (corresponding to the many minima in the energy landscape) in addition to the physical solution (s_- in the case of an increasing H), i.e.,

$$s^{\rm sol}(\mathbf{x},t) = s_-(\mathbf{x},t) + s'(\mathbf{x},t) + s''(\mathbf{x},t) + \cdots. \tag{32}$$

Hence the action

$$S=\int dt d^dx \hat{s}(-\eta\partial_t s + K\nabla^2 s + H + as + bs^3 + \cdots) + \int dt dt' d^dx \frac{R}{2}\hat{s}(\mathbf{x},t)\hat{s}(\mathbf{x},t') \tag{33}$$

does not describe the magnetization, unless it is corrected in such a way as to remove the unphysical solutions. These are inopportunely introduced through Eq. (9), which should in fact be written as

$$\int \mathcal{D}s\,\delta\left(-\eta\partial_t s + K\nabla^2 s + H + h - \frac{\partial \tilde{V}}{\partial s}\right)\times(\text{Jacobian}) = \int \mathcal{D}s\{\delta[s-s_-(\mathbf{x},t)] + \delta[s-s'(\mathbf{x},t)] + \delta[s-s''(\mathbf{x},t)] + \cdots\}. \tag{34}$$

Therefore, if we substract the quantity

$$\overline{\delta(s-s') + \delta(s-s'') + \cdots}$$

from the functional e^S, we obtain a well-defined theory, which incorporates only the physical solution and yields all the correct correlation and response functions. Although this method is impossible to implement (since it explicitly involves the unphysical solutions), a restricted, weaker version is applicable, and fully serves our purposes. The investigation of the critical behavior and the calculation of the corresponding exponents relies primarily on correlation and response functions, namely the field density s and its susceptibility $\partial s/\partial t$, *which are linear in s*. For such objects, $\delta(s-s_-)+\delta(s-s')+\delta(s-s'')+\cdots$ can be replaced with $\delta(s-s_- -s'-s''-\cdots)$. Furthermore, only averaged quantities are of interest to the study of the critical point, and linearity allows us to perform the average inside the argument of the δ function, leading to

$$\delta[s-m_-(t)-\sigma(t)], \tag{35}$$

where

$$\sigma(t) = \overline{s'(\mathbf{x},t) + s''(\mathbf{x},t) + \cdots}. \tag{36}$$

A comparison with Eqs. (9) and (16) implies that a weight e^S which properly describes the averaged theory is obtained by adding $\sigma(t)$ to the argument of the action. Indeed, for

$$\langle s\rangle \equiv \int \mathcal{D}\hat{s}\mathcal{D}s\, s(\mathbf{x},t) e^{S[\hat{s},s]}, \tag{37}$$

we have $\langle s\rangle = m_-(t)+\sigma(t)$; therefore, shifting the field by $\sigma(t)$ yields

$$\int \mathcal{D}\hat{s}\mathcal{D}s\, s(\mathbf{x},t) e^{S[\hat{s},s+\sigma]} = \int \mathcal{D}\hat{s}\mathcal{D}s[s(\mathbf{x},\mathbf{t})-\sigma(t)]e^{S[\hat{s},s]} = \langle s\rangle - \sigma(t) = m_-(t). \tag{38}$$

The average spin dynamics is thus properly described by the corrected action

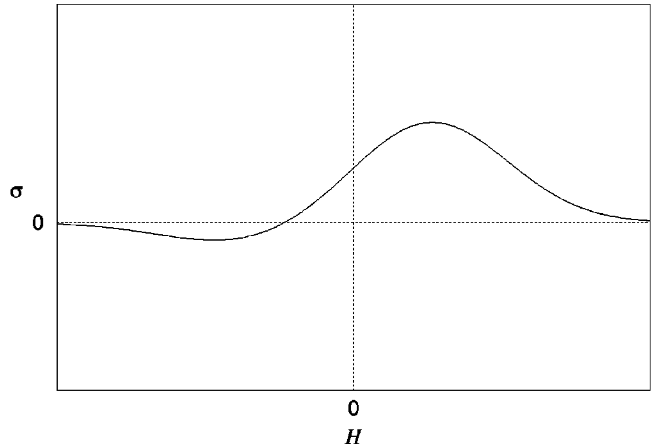

FIG. 3. Schematic shape of the parameter σ as a function of the magnetic field.

$$S=S[\hat{s},s+\sigma]$$
$$=\int dtd^dx\hat{s}(-\eta\partial_t s+K\nabla^2 s+H+A_0+A_1 s+A_2 s^2+A_3 s^3+\cdots)+\int dtdt'd^dx\frac{R}{2}\hat{s}(\mathbf{x},t)\hat{s}(\mathbf{x},t'), \qquad (39)$$

where

$$A_0=-\eta\partial_t\sigma+a\sigma+b\sigma^3+\cdots,$$
$$A_1=a+3b\sigma^2+\cdots,$$
$$A_2=3b\sigma+\cdots, \qquad (40)$$
$$A_3=b+\cdots,$$
$$\vdots\,.$$

Although we cannot obtain the precise form of $\sigma(t)$ without solving for the many minima, its qualitative shape is easily found. Consider a very large (positive or negative) magnetic field. The unphysical minima are due only to the spins in a very large random field h, such that the potential they feel still has two minima. When we take the average, the minima $s'+s''+\cdots$ contribute to σ only with a very small weight $[\propto\exp(-h^2/2R)]$, implying

$$\sigma\rightarrow 0^{\pm}\quad\text{as}\quad H\rightarrow\pm\infty. \qquad (41)$$

Furthermore, it follows from the definition of σ that $\sigma=-m_-$ at $H=0$. Thus, $\sigma(H)$ has a bell shape, shifted to the right since the sum of σ and m_- is an odd function of H (Fig. 3). The parameter $\sigma(t)$, which corrects the coefficients in S, breaks the up-down symmetry present in the action of Eq. (33). This is physically required, since in a hysteretic system, the (nonequilibrium) magnetization breaks that symmetry, in particular at $H=0$.

To discuss the relevant terms in the action, we apply a renormalization transformation in $6-\epsilon$ dimensions. All terms of the action higher than cubic in s are irrelevant, and after a large enough number of coarse-graining steps, we are left with the renormalized action

$$S=\int dtd^dx\hat{s}(-\eta\partial_t s+K\nabla^2 s+\tilde{A}_0+\tilde{A}_1 s+\tilde{A}_2 s^2+\tilde{A}_3 s^3)+\int dtdt'd^dx\frac{R}{2}\hat{s}(\mathbf{x},t)\hat{s}(\mathbf{x},t'), \qquad (42)$$

where $\tilde{A}_{i\geqslant 1}$'s are functions of H through σ and $\tilde{A}_3$ is negative to prevent the spins from diverging in magnitude. Furthermore, it is easily shown that $\tilde{A}_i$ is an even (odd) function of σ for i odd (even). In particular, since $\sigma\rightarrow 0$ as $H\rightarrow\pm\infty$, we also have that $\tilde{A}_2\rightarrow 0$ as $H\rightarrow\pm\infty$. Now, in order to find the critical point, let us expand the field about some value $\mu(t)$, so as to cancel $\tilde{A}_2$. With the choice $\mu=-\tilde{A}_2/3\tilde{A}_3$, the action in terms of $s'\equiv s-\mu$ becomes

$$S=\int dtd^dx\hat{s}(-\eta\partial_t s'+K\nabla^2 s'+\tilde{A}_0'+\tilde{A}_1's'+\tilde{A}_3's'^3)+\int dtdt'd^dx\frac{R}{2}\hat{s}(\mathbf{x},t)\hat{s}(\mathbf{x},t'), \qquad (43)$$

where the coefficients have been corrected by μ. Since $\langle s\rangle\rightarrow\pm\infty$ as $H\rightarrow\pm\infty$, $\langle s\rangle$ crosses μ at a given H_0, i.e., $\langle s(H_0)\rangle=\mu(H_0)$. Hence $\langle s'(H_0)\rangle=0$, which implies in general that $\tilde{A}_0'(H_0)=0$. The action thus reduces to that studied in Refs. [1–3], and is critical for a specific amount of disorder R.

C. The vector model

1. The appropriate action

The first question in the case of vector spins is whether the action of Eq. (17) correctly describes the system. The single-spin problem is identical to that of a bead moving in a Mexican hat tilted by $\mathbf{H}+\mathbf{h}$, the sum of the external $\mathbf{H}(t)$ and the quenched random field $\mathbf{h}$. The bead simply turns around the bump of the hat, i.e., the spin always points in the direction of $\mathbf{H}+\mathbf{h}$, as in the equilibrium problem. The motion is governed by a single minimum, and there is no hysteresis. Similarly, mean-field theory reduces the time evolution of the spin field to that of a single degree of freedom, and yields no hysteresis for any value of the disorder. These observations may suggest that Eq. (17) properly describes the problem, and that criticality occurs at $\mathbf{H}=\mathbf{0}$, simultaneously for the components of the field parallel and perpendicular to $\mathbf{H}$. However, although a single spin has only one minimum in its energy landscape, a configuration of several spins may have many. Consider a system composed of two spins of unit length, in an increasing magnetic field $\mathbf{H}$, and with random fields $+\boldsymbol{\Delta}$ and $-\boldsymbol{\Delta}$ perpendicular to $\mathbf{H}$, as illustrated in Fig. 4.

The Hamiltonian for this system is

$$\mathcal{H}=-J\mathbf{s}_1\cdot\mathbf{s}_2-\boldsymbol{\Delta}\cdot(\mathbf{s}_1-\mathbf{s}_2)-\mathbf{H}\cdot(\mathbf{s}_1+\mathbf{s}_2). \qquad (44)$$

If θ_1 and θ_2 are the angles of the spins with respect to $\mathbf{H}$, the minimum energy path that the spins follow imposes $\theta_1=-\theta_2\equiv\theta$, and in terms of θ, the energy is

$$\mathcal{H}=2\{J(\sin\theta)^2-\Delta\sin\theta+H\cos\theta\}-J. \qquad (45)$$

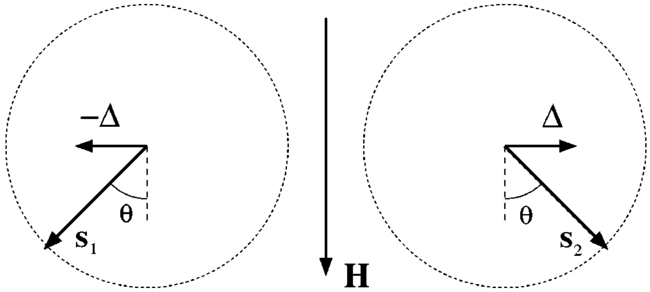

FIG. 4. Illustration of the two-spin toy model described in the text.

For $\Delta < 2J$ and $H = 0$, the energy as a function of θ is a symmetric double well centered on $\theta = \pi/2$ (Fig. 5), similar to the scalar single-spin energy landscape, with two minima at $\sin\theta = \Delta/2J$. A nonvanishing H tilts the double well, and ultimately suppresses one of the two minima. The ferromagnetic interaction causes the two spins to pull each other back before jumping ahead, thereby investing the n-component field's motion with a hysteretic, uniaxial-like character. That is, the two-spin toy system as a whole goes over an energy barrier, reminiscent of the motion of a scalar spin and in contrast with that of a single multicomponent spin which turns around the energy barrier. Interestingly, the Ising (hysteretic) behavior of the *longitudinal* component results both from the interaction and from the presence of a *transverse* random field.

In the zero dimensional (single spin) or infinite dimensional (mean field) case, the system follows a single minimum and there is no hysteresis. When fluctuations are included, however, the above toy system shows that many minima appear, allowing for a hysteretic behavior. Thus, as in the scalar case, the action of Eq. (17) has to be corrected in order to remove the unphysical minima. Following the procedure of Sec. IV B, we replace $\mathbf{s}(\mathbf{x},t)$ by $\mathbf{s}(\mathbf{x},t)+\sigma(t)$, to obtain, if σ is parallel to $\mathbf{H}$, the corrected action

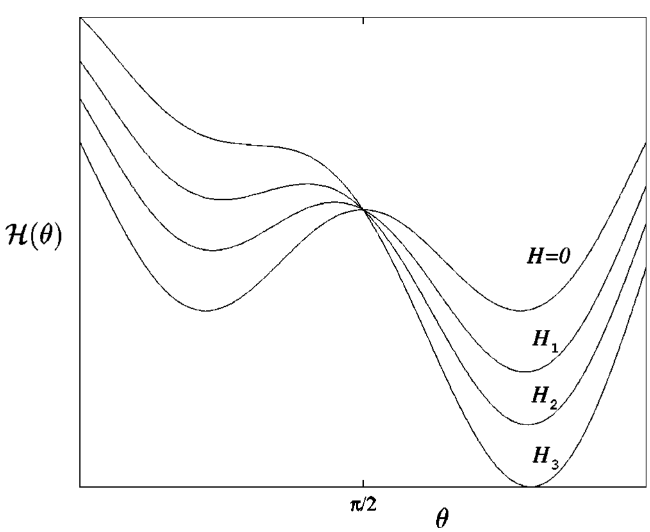

FIG. 5. Energy profile of the two-spin toy model, for different values of the magnetic field H. At $H=0$, the double well is symmetric. Larger values of H tilt the curve, eventually suppressing the left minimum.

$$
\begin{aligned}
S &= S[\hat{\mathbf{s}},\mathbf{s}+\sigma] \\
&= \int dt d^d x \hat{s}_\parallel \{ -\eta \partial_t (s_\parallel + \sigma) + K\nabla^2 s_\parallel \\
&\quad + H + c_1(s_\parallel+\sigma) + c_2[(s_\parallel+\sigma)^2 + \mathbf{s}_\perp^2](s_\parallel+\sigma) + \cdots \} \\
&\quad + \int dt dt' d^d x \frac{R}{2} \hat{s}_\parallel(\mathbf{x},t)\hat{s}_\parallel(\mathbf{x},t') + \int dt d^d x \hat{\mathbf{s}}_\perp \cdot \{ -\eta\partial_t \mathbf{s}_\perp \\
&\quad + K\nabla^2 \mathbf{s}_\perp + c_1 \mathbf{s}_\perp + c_2[(s_\parallel+\sigma)^2+\mathbf{s}_\perp^2]\mathbf{s}_\perp + \cdots \} \\
&\quad + \int dt dt' d^d x \hat{\mathbf{s}}_\perp(\mathbf{x},t)\cdot \hat{\mathbf{s}}_\perp(\mathbf{x},t'),
\end{aligned} \tag{46}
$$

where the subscripts $\parallel$ and $\perp$ refer to the components parallel and perpendicular to the magnetic field. Expanding the polynomials and absorbing the parameter σ in thereby corrected coefficients leads to the form

$$
\begin{aligned}
S = &\int dt d^d x \hat{s}_\parallel (-\eta\partial_t s_\parallel + K\nabla^2 s_\parallel + A_0 + A_1 s_\parallel + A_2 s_\parallel^2 + \overline{A}_2 \mathbf{s}_\perp^2 \\
&+ A_3 s_\parallel^3 + \overline{A}_3 \mathbf{s}_\perp^2 s_\parallel + \cdots) + \int dt dt' d^d x \frac{R}{2} \hat{s}_\parallel(\mathbf{x},t)\hat{s}_\parallel(\mathbf{x},t') \\
&+ \int dt d^d x \hat{\mathbf{s}}_\perp \cdot (-\eta\partial_t \mathbf{s}_\perp + K\nabla^2 \mathbf{s}_\perp + B_1 \mathbf{s}_\perp + B_2 s_\parallel \mathbf{s}_\perp \\
&+ B_3 \mathbf{s}_\perp^2 \mathbf{s}_\perp + \overline{B}_3 s_\parallel^2 \mathbf{s}_\perp + \cdots) \\
&+ \int dt dt' d^d x \frac{R}{2} \hat{\mathbf{s}}_\perp(\mathbf{x},t)\cdot\hat{\mathbf{s}}_\perp(\mathbf{x},t').
\end{aligned} \tag{47}
$$

Clearly, only even powers of $s_\perp$ are present in the first line, while only odd powers occur in the third. A term of order p in $\mathbf{s}$, trivially (i.e., ignoring the coarse-graining step which couples different order terms) scales with $b^{2+d}\hat{\zeta}\zeta^p = b^{2p-(p-1)d/2}$. In dimensions $d \leqslant 4$, all terms are relevant, and a nontrivial fixed point at long length scales cannot in general be reached by tuning merely two quantities, namely the external magnetic field and the amount of randomness. It is much the same for $d=5$, in which terms up to $p=5$ are relevant. In $6-\epsilon$ dimensions, however, all terms with $p>3$ are irrelevant under the RG, leading to an effective action of the form of Eq. (47), with any terms not displayed set to zero. Including the corrections due to coarse graining, to lowest nontrivial order (in the interaction and in $\epsilon = 6-d$) the recursion relations for the remaining 10 vertices read

$$A_0' = b^{3-\epsilon/2}[A_0 + (I+2A_1)A_2 + (n-1)(I+2B_1)\overline{A}_2],$$

$$A_1' = b^2[A_1 + 3(I+2A_1)A_3 + (n-1)(I+2B_1)\overline{A}_3],$$

$$B_1' = b^2[B_1 + (n+1)(I+2A_1)B_3 + (I+2B_1)\overline{B}_3],$$

$$A_2' = b^{1+\epsilon/2}[A_2 + 18A_2A_3 + 2(n-1)\overline{A}_2\overline{B}_3 + 2(n-1)B_2\overline{A}_3],$$

$$\overline{A}_2' = b^{1+\epsilon/2}[\overline{A}_2 + 2A_2\overline{A}_3 + 2(n+1)\overline{A}_2 B_3 + 2B_2\overline{A}_3 + 4\overline{A}_2\overline{A}_3],$$

$$B_2' = b^{1+\epsilon/2}[B_2 + 4B_2\overline{B_3} + 2(n+1)B_2B_3 + 2B_2\overline{A_3} + 4A_2\overline{B_3} + 4\overline{A_2}\,\overline{B_3}],$$

$$A_3' = b^{\epsilon}[A_3 + 18A_3^2 + 2(n-1)\overline{A_3}\,\overline{B_3}],$$

$$\overline{A_3}' = b^{\epsilon}[\overline{A_3} + 4\overline{A_3}^2 + 6A_3\overline{A_3} + 2(n+1)\overline{A_3}B_3 + 4\overline{A_3}\,\overline{B_3}],$$

$$B_3' = b^{\epsilon}[B_3 + 2(n+7)B_3^2 + 2\overline{A_3}\,\overline{B_3}],$$

$$\overline{B_3}' = b^{\epsilon}[\overline{B_3} + 4\overline{B_3}^2 + 2(n+1)B_3\overline{B_3} + 6A_3\overline{B_3} + 4\overline{A_3}\,\overline{B_3}], \quad (48)$$

where $I = \Lambda^2(b^2-1)/2b^2 \ln b$ and a factor of $R \ln b/(4\pi)^3$ is absorbed in a redefinition of A_2, $\overline{A_2}$, B_2, A_3, $\overline{A_3}$, B_3, and $\overline{B_3}$. [For the derivation of the corresponding relations for the scalar model, the reader is referred to Refs. [2,3]; the extension to the effective action of Eq. (47) is straightforward.]

2. Ising criticality

Clearly, a nontrivial fixed point for all 10 vertices cannot be reached in general if only two quantities are to be tuned. However, if R is appropriately tuned, A_1 flows to its fixed (finite) value while B_1 grows indefinitely. Then, under the RG, the interactions in the perpendicular components become less and less important with respect to the quadratic term. After sufficient rescalings, the theory becomes Gaussian in the perpendicular fields which can be integrated out, resulting in an effective action for $s_\parallel$ identical to the scalar action of Eq. (39). Thus, the critical point of our $O(n)$ model is generically described by the same action as in Refs. [1–3], yielding identical recursion relations and exponents. This can be physically understood in the following way: if a configuration of several spins gives rise to many minima, a coarse-grained vector-spin system roughly looks like an Ising system, leading to scalarlike critical behavior. The latter occurs for a nonvanishing value of the magnetic field or the magnetization, at which the full rotational symmetry of the $O(n)$ model is broken.

3. Transverse criticality

In the case considered above, the longitudinal field is massless and the transverse components are massive. Alternatively, one should be able to reach another fixed point which incorporates the reverse situation. In Eq. (47), let us expand the longitudinal field about some parameter $\lambda(t)$, so chosen as to cancel the linear term A_0, i.e., we write

$$s_\parallel = s_\parallel' + \lambda, \quad (49)$$

with λ satisfying

$$A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 = 0. \quad (50)$$

The theory is then Gaussian in $s_\parallel'$. Integrating out the longitudinal field, the action *for the transverse field* reduces to Eq. (42) with $(\hat{\mathbf{s}}_\perp, \mathbf{s}_\perp)$ instead of $(\hat{s}, s)$, and $\tilde{A}_0 = \tilde{A}_2 = 0$, i.e.,

$$S = \int dt d^d x \hat{\mathbf{s}}_\perp \cdot [(-\eta\partial_t + K\nabla^2 + r_\perp)\mathbf{s}_\perp + B_3 \mathbf{s}_\perp^2 \mathbf{s}_\perp], \quad (51)$$

where $r_\perp$ denotes the corrected mass (see below). As in the case of the scalar model, this action is critical for an appropriate value of the disorder, or equivalently of the ''mass.'' The recursion relations for $r_\perp$ and $u_\perp \equiv R[\ln b/(4\pi)^3]B_3$ can be read off from Eq. (48) by setting $A_i = \overline{A_i} = B_2 = \overline{B_3} = 0$, as

$$r_\perp' = b^2\{r_\perp + [(n-1)+2](I + 2r_\perp)u_\perp\},$$
$$u_\perp' = b^{\epsilon}\{u_\perp + 2[(n-1)+8]u_\perp^2\}. \quad (52)$$

From these recursion relations we can obtain the static exponent ν with which the correlation length ξ diverges. By definition $\xi \sim (R-R_c)^{-\nu} \sim |r-r_c|^{-\nu}$, which, along with $\xi' = \xi/b$ and $(\partial r'/\partial r)_{\text{fixed point}} = b^{2-\{[(n-1)+2]/[(n-1)+8]\}\epsilon} \equiv b^{y_r}$, yields $(b^{y_r}|r-r_c|)^{-\nu} = |r-r_c|^{-\nu} b^{-1}$, i.e., $\nu y_r = 1$, and

$$\nu = \frac{1}{2} + \frac{1}{2}\frac{(n-1)+2}{(n-1)+8}\epsilon. \quad (53)$$

Note that the action of Eq. (51) is identical to the critical action for $n-1$ (weakly coupled) scalar fields, with rotational symmetry. It follows immediately from this consideration that the recursion relations and exponents for the longitudinal (Ising-like) fixed point are identical to those calculated here, with $n-1=1$.

In the reduced action, the transverse components' bare mass becomes

$$r_\perp = B_1 + B_2\lambda + \overline{B_3}\lambda^2. \quad (54)$$

The critical line $r_\perp^*(u_\perp)$ is in the lower half $(r_\perp < 0)$ of the $(r_\perp, u_\perp)$ plane. Since $A_0 \to \pm\infty$ as $H \to \pm\infty$, Eq. (50) implies that λ also goes from $-\infty$ to $+\infty$ as the field is increased (similarly to $\tilde{A}_3 < 0$ [Eq. (42)], we have $\overline{B_3} < 0$). Thus, $r_\perp$ crosses the critical line for a given value of H, provided that $B_2^2 - 4\overline{B_3}[B_1 - r_\perp^*(u_\perp)] \geq 0$, i.e., unless B_1 is too negative (for a Mexican hat potential, we expect $B_1 > 0$). This may not be true if A_0 is a discontinuous function of H. But $A_0(H)$ is clearly continuous for critical and higher disorders. (A very high disorder, though, may suppress this transverse criticality by renormalizing $r_\perp$ into large negative values.) Whether or not the fixed point controlling the transverse criticality is reachable experimentally depends on what region of the space of the theory's parameters is swept when the physical quantities at hand in the experiment are varied. Unlike many equilibrium problems in which the symmetry is broken by an infinitesimal field, criticality can occur here at large values of H, allowing for non-negligible higher-order terms, such as $H^2 s_\parallel^2$ or $H^2 s_\parallel^2 \mathbf{s}^2$, in the Hamiltonian. These terms appear in the theory as modified (strong) functional dependences of the coefficients $A_i, \overline{A_i}, B_i, \overline{B_i}$, on the magnetic field, which, depending on the trend, may favor a transverse instability.

The transverse critical point corresponds to an infinite susceptibility at some $H_\perp$, resulting in a spontaneous *trans-*

verse magnetization. A similar phenomenon was noted for the case of a pure (thermal) system in Ref. [16]. There, however, the appearance of a transverse magnetization is due to a magnetic field which oscillates at high frequency. It is a purely dynamical effect, not observed in the $\eta \to 0$ limit. In our case, the effect is due to the presence of a quenched randomness, and its nonequilibrium nature lies in the metastability of the minima occupied by the system during its history. This transverse instability, though, differs from the longitudinal criticality in that it occurs for a range of disorders, rather than at a specifically tuned amount of randomness. Its underlying physical mechanism may, for example, be illustrated by two beads, attached to each other by a spring, and flowing on the two sides of a Mexican hat's bump, as it is progressively tilted. At some point, it might become favorable for one of the beads to jump on the opposite side of the rim, and to continue its motion next to the other bead, corresponding to a transverse ordering.

At $H_\perp$, the transverse components choose one of the many minima, which breaks the rotational symmetry in the $(n-1)$-dimensional transverse space. We therefore have to correct the action of Eq. (47) for $H > H_\perp$, as we did for the longitudinal field. This, however, does not alter the analysis of the longitudinal criticality. By shifting $\mathbf{s}_\perp$, we eliminate the transverse linear term and can then follow the same procedure as before. We used the fact that the correction to Eq. (47) vanishes for $H \to \pm\infty$, and this clearly still holds here.

4. $O(n)$ criticality

In the above, we examined two distinct fixed points, corresponding to an infinite longitudinal susceptibility and a transverse ordering, respectively. For a specific choice of the theory's parameters, the two should merge into a single, rotationally invariant (in the *n*-dimensional space of the fields) fixed point, at which all components become simultaneously critical. In addition to the magnetic field and disorder, how many quantities should we tune to reach such an $O(n)$ *fixed point*? While Eq. (47) is a suitable formulation of the model when the symmetry is broken in the longitudinal direction, the form of Eq. (46) is more appropriate close to a fully rotationally invariant theory. An RG transformation can be carried out for it, as was done for the action of Eq. (47). Since $\sigma(t)$ is spatially uniform, only its $\mathbf{q}=\mathbf{0}$ mode is nonvanishing. The parameter σ, therefore, does not participate in the coarse-graining transformation and its normalization is given by the rescalings of the coordinates and the fields as

$$\sigma'(t) = \zeta^{-1}\sigma(b^2 t) = b^{d/2-2}\sigma(b^2 t). \tag{55}$$

The recursion relations for H, c_1, and c_2, on the other hand, are obviously given by Eq. (48), with $A_0 = H$, $A_1 = B_1 = c_1$, $A_3 = \underline{A}_3 = B_3 = \underline{B}_3 = c_2$, and $A_2 = \underline{A}_2 = B_2 = 0$.

By tuning both the magnetic field H and H_0, defined as the zero of σ $[\sigma(H_0)=0]$, to zero, the action of Eq. (46) reduces to the form of Eq. (51), with n fields $\mathbf{s}$ rather than the $n-1$ components of $\mathbf{s}_\perp$, which is critical for a given value of the disorder. Consequently, the recursion relations and exponents are identical to those characterizing the transverse critical point, but with n replacing $n-1$. Furthermore, since $H=0$ and the action is fully rotationally invariant, hysteresis is suppressed at the $O(n)$ critical point; in particular, $\mathbf{m}(H=0)=0$. Finally, by identifying the relevant parameter σ, we have established that only a single additional physical quantity needs to be tuned to reach the $O(n)$ fixed point (provided that H_0 crosses zero as the physical quantity in question is varied, at $H=0$).

The symmetric fixed point is characterized by three relevant parameters, namely r and H, with the usual exponents $y_r = 2-\epsilon(n+2)/(n+8)$ and $y_H = 3-\epsilon/2$, and σ with $y_\sigma = d/2-2$, to $O(\epsilon)$. The parameter σ is a measure of the deviation from a symmetric (nonhysteretic) theory at $H=0$, and, according to the above exponents, is much less relevant than the other symmetry-breaking term H, or of the mass r, to $O(\epsilon)$. Reference [4] notes the fact that the "critical region" is unusually large in the scalar model, as signaled by power laws with surprisingly high cutoffs even a few percent to one hundred percent away from the critical disorder, and briefly discusses possible origins of this phenomenon. This observation, along with the weakness of σ's relevance, suggests in our case that even away from the $O(n)$ fixed point, an $O(n)$-like behavior might be displayed at short ranges by the system, before crossing over to an Ising-like behavior at long enough time and length scales [9].

As mentioned in the Introduction, a vanishing magnetization does not *a priori* ensure a fully rotationally invariant system; higher moments could in general display anisotropy. In that case, a complete theory would break the rotational symmetry, leading to anisotropic exponents different from y_r, y_H, and y_σ above, which are the outcome of a restricted action that properly describes only quantities linear in the spin field. In general we may ask the following question, which applies to the Ising, transverse, and $O(n)$ cases. Would a complete theory yield the same exponents as our resticted theory? The latter correctly describes the magnetization $\mathbf{m}_-(t)$. That is, if one were able to calculate the path integral

$$\int \mathcal{D}\hat{s}\mathcal{D}s\,\mathbf{s}(\mathbf{x},t)e^{S[\hat{\mathbf{s}},\mathbf{s}+\sigma]}, \tag{56}$$

one would obtain $\mathbf{m}_-$ as a function of $\mathbf{H}(t)$, R, and an additional tuning parameter (corresponding to H_0), and thus the associated exponents describing the critical singularity of the magnetization. It is easy to see that, as for the nonrandom equilibrium thermal Landau-Ginzburg model, the exponents y_r, y_H, and y_σ have a one-to-one correspondance with the exponents that characterize the singular behavior of the magnetization, as well as with those associated with a number of other quantities, such as the correlation length. This immediately implies an affirmative answer to the above question; the requirement that the action should describe the physical magnetization is a strong enough constraint for the theory to correctly generate the singularity of, e.g., the correlation length. Evidently, this argument does not apply to the critical behavior of higher moments of the spin field distribution, and whether or not the latter are isotropic at criticality is an interesting open question. In fact, even the isotropy of y_r at the $O(n)$ critical point may seem surprising at first, since the history of the system apparently sets up an anisotropic context for the spins' motion. It is consistent, however, with the fact noted in Sec. IV A that different time slices of the action decouple under the RG in the $\eta\to 0$ limit, in which the static exponents are calculated.

As the reader may have noticed, our recursion relations and exponents in $d=6-\epsilon$ are none other than the recursion relations and exponents for the pure (equilibrium) $O(n)$ model in $d=4-\epsilon$. Indeed, dimensional reduction to two lower dimensions holds perturbatively [2,3,17,15]. This allows us to obtain, with no further calculational effort, the expansion for ν to higher orders in ϵ [18].

The dynamic exponent z, on the other hand, cannot be obtained from a time-independent model. How is the renormalization procedure modified when $\eta\neq 0$? In the coarse-graining process, vertices which couple different times are generated. In particular, a nonlocal quadratic term of the type

$$\int_{-\infty}^{t'} dt \hat{s}_{\parallel}(t') s_{\parallel}(t) f(t'-t) \tag{57}$$

is generated in the action, where f is some function (containing free propagators, disorder, etc). Expanding $s_{\parallel}(t)$ about t', this is rewritten as

$$\int_{-\infty}^{t'} dt \hat{s}_{\parallel}(t') \left\{ s_{\parallel}(t') + \left.\frac{\partial s_{\parallel}}{\partial t}\right|_{t'} (t-t') + \cdots \right\} f(t'-t). \tag{58}$$

The first term contributes to the coarse-grained ''mass,'' the second results in a correction to z. In fact, the perturbative calculation we just briefly described is similar to that done by Krey [19] for the time-dependent thermal random field model, where he calculates z to $O(\epsilon^3)$, giving to lowest nontrivial order in ϵ,

$$z=2+\frac{n+2}{(n+8)^2}\epsilon^2. \tag{59}$$

(Here, n is the number of components that become massless at criticality.)

V. CONCLUSION

The main aim of the present paper was to extend earlier work on critical hysteresis to the case of an ordering of continuous symmetry. Our central result is that, generically, criticality is Ising-like, i.e., the critical exponents calculated for a scalar field model [1–3] still hold for a vector field. By tuning a single additional quantity, however, a fully symmetric fixed point can be reached, to which $O(n)$-like exponents are associated. Furthermore, the possibility of a spontaneous nonequilibrium transverse magnetization is unveiled and examined using a perturbative renormalization scheme. In addition, we have clarified several issues pertaining to the path-integral formalism, in particular to the structure of time dependences, and to the problem of multiple solutions. These analyses are useful for our treatment of the problem, as well as for a clearer understanding of earlier works.

An interesting question which remains open is that of the lower critical dimension. In the fully rotational case, naive dimensional reduction suggests that the lower critical dimension is 4. If this were the case, no vector criticality should be observed in three dimensions. Several scenarios are possible. For example, the hysteresis curve may display a jump for low disorder and be continuous for high disorder, but without the intermediate limit case of a continuous curve with a diverging slope at a given point. A more probable scenario is one in which the hysteresis curve is already smooth for an infinitesimal amount of disorder.

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APPENDIX: EXPANSION ABOUT MEAN FIELD THEORY

In Refs. [1–3], the effective action is written as an expansion about mean-field theory (MFT). In such a formulation, the linear term in the action [corresponding to A_0 in Eq. (47)] is vanishing. The ''masses,'' i.e., the coefficients $r_{\parallel}$ and $r_{\perp}$ [corresponding to A_1 and B_1 in Eq. (47)] of the parallel and perpendicular quadratic terms are calculated as the elements of a particular response tensor within MFT [13,1–3]. In this appendix, we calculate $r_{\parallel}$ and $r_{\perp}$, starting from the mean-field equations of motion, and find that, generically, $r_{\parallel}\neq r_{\perp}$. This appendix thus reaffirms that criticality is Ising-like in general, as found in Sec. IV, and makes contact with the earlier methodology of Refs. [1–3,13].

MFT is defined by an infinite-range coupling, leading to the equations of motion

$$\eta\partial_t \mathbf{s}(\mathbf{x}) = \mathbf{m}(t)+\mathbf{H}(t)+\mathbf{f}(t)+\mathbf{h}(\mathbf{x})+c_1\mathbf{s}(\mathbf{x})+c_2\mathbf{s}(\mathbf{x})^2\mathbf{s}(\mathbf{x}), \tag{A1}$$

where $\mathbf{m}(t)$ is defined self-consistently by $\mathbf{m}(t)=\overline{\mathbf{s}(\mathbf{x},t)|_{\mathbf{f}=0}}$ and $\mathbf{f}(t)$ is a test field. The ''masses'' are the static components of the tensor $\delta\overline{\mathbf{s}}/\delta\mathbf{f}-I$, where I is the identity matrix [13,1–3]. We have

$$\frac{\delta\mathbf{m}}{\delta\mathbf{H}} = \left.\begin{pmatrix} \frac{\delta m_1(H+H_1,H_2,\dots)}{\delta H_1} & \frac{\delta m_1(H+H_1,H_2,\dots)}{\delta H_2} & \cdots \\ \frac{\delta m_2(H+H_1,H_2,\dots)}{\delta H_1} & \frac{\delta m_2(H+H_1,H_2,\dots)}{\delta H_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\right|_{H_1=H_2=\cdots=0} \equiv \begin{pmatrix} \chi_{\parallel}(H) & 0 & 0 & \cdots \\ 0 & \chi_{\perp}(H) & 0 & \cdots \\ 0 & 0 & \chi_{\perp}(H) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{A2}$$

(We consider $\eta\rightarrow 0$, so responses are nonvanishing only at equal times, and only the static part remains. Also, in MFT, the magnetization is always parallel to the magnetic field, leading to vanishing off-diagonal elements and to the relation $\chi_\perp = m/H$ [8].) In order to calculate the relevant tensor, let us split the field **f** between **m** and **H**, in such a way as to satisfy Eq. (A2) for effective magnetization and magnetic field. That is, we write the equations of motion in the following fashion, with corrected **m** and **H**, and with no additional field,

$$\eta\partial_t s_\parallel = \left(m + \frac{\chi_\parallel}{1+\chi_\parallel} f_\parallel\right) + \left(H + \frac{1}{1+\chi_\parallel} f_\parallel\right) + h_\parallel + c_1 s_\parallel + c_2 \mathbf{s}^2 s_\parallel \tag{A3}$$

for the longitudinal component, and

$$\eta\partial_t \mathbf{s}_\perp = \left(\frac{\chi_\perp}{1+\chi_\perp}\mathbf{f}_\perp\right) + \left(\frac{1}{1+\chi_\perp}\mathbf{f}_\perp\right) + \mathbf{h}_\perp + c_1 \mathbf{s}_\perp + c_2 \mathbf{s}^2 \mathbf{s}_\perp \tag{A4}$$

for the transverse components. Whence

$$\frac{\delta \overline{\mathbf{s}}}{\delta \mathbf{f}} = \begin{pmatrix} \frac{1}{1+\chi_\parallel^{-1}} & 0 & 0 & \cdots \\ 0 & \frac{1}{1+\chi_\perp^{-1}} & 0 & \cdots \\ 0 & 0 & \frac{1}{1+\chi_\perp^{-1}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{A5}$$

and

$$r_\parallel = -\frac{1}{1+\chi_\parallel}, \qquad r_\perp = -\frac{1}{1+\chi_\perp} = -\frac{1}{1+m/H}; \tag{A6}$$

thus, $r_\parallel(H) \neq r_\perp(H)$ in general.

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[9] As in the equilibrium case, the disorder-induced criticality is not affected by thermal fluctuations. Calculationally, this can be detailed as follows. A nonzero temperature is implemented by a stochastic term, analogous to the quenched randomness term, in the equation of motion. When we reformulate the model as a path integral, two similar terms are produced, corresponding to thermal and quenched disorder, respectively. Unlike the latter, the former is uncorrelated both in space *and in time*. As a consequence, the two $\hat{s}$ fields occur at equal time in the temperature-induced term, which is thereby less relevant than its (marginal) disorder-induced counterpart. Thus, thermal fluctuations do not modify the critical behavior, unless they change the symmetries of the problem. Indeed, a nonvanishing temperature will in general displace the critial point in parameter space. If, in the $\eta\rightarrow 0$ limit, thermal fluctuations allow the system to equilibrate for each value of the exterior magnetic field, the magnetization history is nonhysteretic, regardless of the amount of disorder. (If η is nonvanishing, the lag in the response rounds off the crititcal point, leading to a finite dynamic susceptibility.) The absence of hysteresis clearly does not modify the predictions of the exponents in the scalar case. In the case of a multicomponent field, however, the scalar and transverse criticalities would not be at experimental reach, enforcing $O(n)$-like exponents.

[10] To be more accurate, the spins not only follow their local minima, but also precess. However, since H can be increased arbitrarily slowly, the precession time scale can be separated as much as we want from the ''Ω^{-1}'' time scale. Then, considering the motion on an intermediate time scale, we find that the precession averages out, and we are effectively left with the above Langevin equation.

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[12] The path integral is defined as the limit of a time-discretized formulation. In the simplest (Ito) discretization scheme in which the response to a force at time t occurs at time $t+\epsilon$, the Jacobian is clearly independent of the spin field s. In this case, the contraction of $\hat{s}$ and s (response function) vanishes at equal time. Other discretization procedures lead to more complicated Jacobians, which, in all cases, cancel the naive equal-time responses resulting from the discrete equations. (See also [13] and references therein.)

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corresponding minimum. The $\eta \to 0$ limit suppresses the lag in the response, but does not overthrow causality. The latter is ensured by the prescription that a solution should not jump from one minimum to another before the first one disappears. Then, a single minimum is visited if we impose that the solutions, incorporated by the generating functional, exist from $t = -\infty$ to $t = +\infty$. However, if we allow solutions to start or end at finite times, and such solutions *are picked* by the δ function in Eq. (9), many minima are simultaneously included in the path-integral formulation of the theory.
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